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Local Approximation Properties of Spline Projections

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1. INTRODUCTION

Our purpose here is to study local approximation properties of projections onto spline spaces. Given a sequence of spline spaces $\{S_N\}$ of fixed degree $k-1$ and a sequence of projections $\{P_N\}$, $P_N\colon C[0,1] \to S_N$, such that $\| P_N f - f\|_\infty \to 0$ for all $f \in C[0,1]$, we shall say that $\{P_N\}$ approximates well locally or has good local approximation properties if for any $0 \leqslant a < \alpha < \beta < b \leqslant 1$ there exist constants K_1 and K_2 and an integer N_0 such that for all $f \in C[0,1]$,

$$\| P_N f - f\|_{L_\infty[\alpha,\beta]} \leqslant K_1\{\inf_{s\in S_N} \|f - s\|_{L_\infty[a,b]} + K_2(\bar{\Delta}_N)^k \inf_{s\in S_N} \|f - s\|_{L_\infty[0,1]}\} \tag{1.1}$$

for $N \geqslant N_0$. This problem has been studied in the cases of quadratic spline interpolation at the midpoints of mesh intervals and cubic spline interpolation at mesh points; cf. [5, 9, 10]. In the L_2-norm for uniform partitions, this problem has been studied for the least-squares projection by Nitsche and Schatz [13]. For the least-squares projection in the uniform norm, a result of the form (1.1) is implicit in the paper of Douglas, Dupont, and Wahlbin [8] for quasi-uniform partitions. Our approach has been inspired by this latter paper; in particular, by the observation of [8] that if a sequence of positive numbers $\{a_n\}_{n\geqslant 1}$ satisfies $\sum_{j>n} a_j \leqslant M\, a_n$ for some constant M and all n, then $a_j \leqslant K r^j a_1$ for some $K > 0$ and $0 < r < 1$. The main result here is that if the P_N's are locally determined (cf. (3.1)) and if they satisfy some natural uniformity condition (cf. (3.7)), then (1.1) holds (Theorem 3.6). Most, if not all, of the widely studied spline projections satisfy our conditions.

In Section 2, we show that "most" uniformly bounded sequences of projections do not approximate well locally. In Section 3 we give sufficient conditions for (1.1) to hold. In Section 4, we give a few applications and derive some known results. We also find a class of spline projections bounded

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on L_q for some $1 \leqslant q < \infty$ whose L_∞ norms are of the same magnitude as their L_q norms. This enables us to recover the result of [8].

For a partition $\Delta: 0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{N+k-1} = 1$, with $x_i < x_{i+k}$ for some fixed integer $k \geqslant 2$, we define the spline space $S(k, \Delta)$ to be the linear span of the normalized B-splines $\{B_i\}_{i=0}^{N-1}$, where

$$B_i(x) = g(x_{i+1}, \ldots, x_{i+k}; x) - g(x_i, \ldots, x_{i+k-1}; x)$$

and $g(s; t) = (s - t)_+^{k-1}$; cf. [4]. As is well known, there is a constant $D_k > 0$ such that for any such Δ and for any scalars $\{a_i\}$

$$D_k^{-1} \max |a_i| \leqslant \left\| \sum a_i B_i \right\|_\infty \leqslant \max |a_i|. \tag{1.2}$$

Here, $\|\cdot\|_\infty$ is the L_∞-norm on [0, 1]. We define $\bar{\Delta} = \max_i \{x_{i+k} - x_i\}$. For convenience we shall often write S instead of $S(k; \Delta)$. It will be understood that all partitions are of the above form and that, unless stated otherwise, k is a fixed but arbitrary integer greater than one.

The dual space of $C[0, 1]$ will be denoted by C^*. If $f \in C[0, 1]$ and $\lambda \in C^*$, the value of λ at f will be denoted by $\langle \lambda, f \rangle$ or, sometimes, by $\lambda(f)$. For $\lambda \in C^*$ and A a subset of [0, 1] we shall write carr $\lambda \subseteq A$ if $\langle \lambda, f \rangle = 0$ for all f which vanish on A. The smallest such set A is called the carrier of λ. The support of a function $g \in C[0, 1]$ is $\operatorname{supp} g = \overline{\{x: g(x) \neq 0\}}$. Given linear functionals $\{\phi_i\}_{i=0}^{N-1}$, linearly independent over S, *the projection determined by* $\{\phi_i\}$ is P where $Pf = s$ if and only if $\langle \phi_i, f - s \rangle = 0 \ \forall i$. Given $\{\lambda_i\}_{i=1}^M \subseteq C^*$ and $\{g_i\}_{i=1}^M \subseteq C[0, 1]$, the notation $T = \sum \lambda_i \otimes g_i$ means that T is determined by the rule $Tf = \sum \lambda_i(f) g_i \ \forall f \in C[0, 1]$; T is a projection if and only if $\langle \lambda_i, g_j \rangle = \delta_{ij}$. The adjoint of T is $T^* = \sum g_i \otimes \lambda_i$ and $\|T\| = \|T^*\|$.

2. A Negative Result

Most of the known projection schemes onto spline spaces are known to enjoy local convergence properties. In the case of least-squares projections onto spline spaces satisfying a global mesh restriction, this is an immediate consequence of the proof given in [8]. In the case of quadratic spline interpolation, it follows from a matrix theoretic argument; cf. [10]. For cubic spline interpolation, a local convergence theorem can be found in [5]; in addition, a matrix theoretic argument can be found in [9]. The approximation methods investigated in [11] automatically have good local approximation properties. On the other hand, the methods studied in [7] and [12] do not approximate well locally (but the associated projections were not uniformly bounded). It is, therefore, reasonable to ask: for what sequences of projections $\{P_N\}$ onto spline spaces $\{S_N\}$ ($P_N: C[0, 1] \to S_N \equiv S(\Delta_N)$ and $\lim_{N\to\infty} \bar{\Delta}_n = 0$) does $\sup_N \|P_N\| < \infty$ imply that the P_N's approximate well locally? We shall presently show that not all such sequences of projections have this nice property. In the next section we shall give some sufficient conditions for $\{P_N\}$ to approximate well locally.

Let Δ_N be the partition of [0, 1] consisting of the points i/N, $0 \leqslant i \leqslant N$, and let $S_N = S(\Delta_N)$. Define the Banach space $X \equiv \{\{T_N\}: T_N: C[0, 1] \to S_N$, T_N bounded, linear, and $\sup_N \| T_N \| < \infty\}$ with $\|\{T_N\}\|_X \equiv \sup_N \| T_N \|$ and the obvious algebraic operations. Let $\mathscr{P} \subseteq X$ consist of all $\{T_N\}$ such that $T_N: C[0, 1] \to S_N$ is onto and $T_N^2 = T_N \, \forall N$.

PROPOSITION 2.1. *Let $\{P_N\} \in \mathscr{P}$ and $\epsilon > 0$. Then, there is an $f \in C[0, 1] \cap C^\infty[0, \frac{1}{2}]$ and an $\{R_N\} \in \mathscr{P}$ such that $\|\{P_N\} - \{R_N\}\|_X \leqslant \epsilon$ and $\overline{\lim}_{N\to\infty} N \cdot \| R_N f - f \|_{L_\infty[0,1/4]} > 0$.*

Proof. Let $P_N = \sum_i \lambda_i^N \otimes B_i^N$ where $\{B_i^N\}$ is the normalized B-spline basis for S_N; we may assume that $\forall f \in C[0, 1] \cap C^\infty[0, \frac{1}{2})$, $\overline{\lim}_{N\to\infty} N \cdot \| P_N f - f \|_{L_\infty[0,1/4]} = 0$. Let

$$f(x) = \begin{cases} e^x, & 0 \leqslant x \leqslant \frac{1}{2}, \\ g(x), & \frac{1}{2} \leqslant x \leqslant 1, \end{cases} \tag{2.1}$$

where $g(\frac{1}{2}) = e^{1/2}$ and g is continuous on $[\frac{1}{2}, 1]$ but has a derivative nowhere on $[\frac{1}{2}, 1]$. Since $\inf\{\| g - s \|_{L_\infty[1/2,1]}: s \in C[\frac{1}{2}, 1] \cap S_N\} \geqslant C/N$ for some $C > 0$ independent of N, there is a bounded linear functional μ_N on $C[0, 1]$ such that $\mu_N(s) = 0 \; \forall s \in S_N$, $\| \mu_N \| = 1$, $\mu_N(g) \geqslant C/N$, and $\mu_N(h) = 0$ if $h \in C[0, 1]$ and $h(x) = 0 \; \forall x \in [\frac{1}{2}, 1]$. Let $Q_N \equiv \epsilon \sum_i \mu_N \otimes B_i^N$ and $R_N \equiv P_N + Q_N$. Now, for $x \in [0, \frac{1}{4}]$, we have

$$\begin{aligned} N \cdot | R_N f(x) - f(x)| &\geqslant N \cdot | Q_N f(x)| - N | P_N f(x) - f(x)| \\ &> \epsilon \cdot C - N | P_N f(x) - f(x)|. \end{aligned}$$

Q.E.D.

The following result further demonstrates the scarcity of sequences of projections with good local approximation properties.

PROPOSITION 2.2. *Let f be as in* (2.1), *let $0 \leqslant z \leqslant \frac{1}{4}$, and let $\epsilon > 0$. Then $A \equiv \{\{P_N\} \in \mathscr{P}: \overline{\lim}_{N\to\infty} N^{1+\epsilon} \mid P_N f(z) - f(z)\mid$ is finite$\}$ is of the first category in $\mathscr{P}$.*

Proof. $A \subseteq \bigcup_{M=1}^\infty A_M$ where $A_M = \{\{P_N\}: | P_N f(z) - f(z)| \leqslant M/N^{1+\epsilon}\}$ Each A_M is clearly closed and the construction used in the previous lemma can be used to show that no A_M has an interior in $\mathscr{P}$; we leave the details to the reader.

While the above results show that "most" sequences of projections do not have nice local approximation properties, it is the case that most (if not all) of the widely used projection schemes do approximate well locally. As we show in the next section, this is essentially a consequence of the uniform boundedness of the sequence of projections.

3. Main Results

Let $P = \sum \lambda_i \otimes B_i$ be a projection determined by $\{\mu_i\}$ where

$$\operatorname{carr} \mu_i \subseteq \operatorname{supp} B_i \ \forall i \qquad \text{and} \qquad \| \mu_i \| \leqslant 1 \ \forall i. \tag{3.1}$$

It follows that $\lambda_i = \sum_j a_{ij} \mu_j$ for some constants a_{ij}. There is a constant $D > 0$ independent of Δ such that for any $f \in C[0, 1]$ and any i and l $(l \geqslant k)$

$$\| f - g \|_{L_\infty[x_i, x_{i+l}]} \leqslant D \inf_{s \in S} \| f - s \|_{L_\infty[x_{i-k}, x_{i-l-k}]}$$

for some $g \in S$; see [2] or [11] for details. Since P is a projection, $Pf - f = P(f - g) + (g - f)$. Therefore, to estimate $Pf - f$, it suffices to estimate $P(f - g)$. Let $x \in [x_l, x_{l+k}]$. Then, for any $m \geqslant 0$,

$$\begin{aligned} | P(f - g)(x) | &= \Big| \sum_{|i-l|<k} \sum_j a_{ij} \mu_j (f - g) B_i(x) \Big| \\ &\leqslant (2k - 1) \max_{|i-l|<2k} \Big\{ \sum_{|j-i| \leqslant m} | a_{ij} | \, | \mu_j (f - g) | \\ &\qquad + \sum_{|j-i|>m} | a_{ij} | \, | \mu_j (f - g) | \Big\}. \end{aligned}$$

By (3.1) we obtain

$$\begin{aligned} \| P(f - g) \|_{L_\infty[x_l, x_{l+k}]} \leqslant (4k - 1) \{ K \| f - g \|_{L_\infty[x_{l-k-m+1}, x_{l+2k+m}]} \\ + \| f - g \|_\infty R(l, m) \}. \end{aligned} \tag{3.2}$$

In this case

$$K = \max_{|i-l|<2k} \sum_{|j-i| \leqslant m} | a_{ij} |$$

and

$$R(l, m) = \max_{|i-l|<2k} \sum_{|j-i|>m} | a_{ij} |.$$

The following result is known, but since it motivates much of what will follow, we include a proof.

PROPOSITION 3.1. *Let $P = \sum \lambda_i \otimes B_i$ be the projection determined by $\{\mu_i\}$ where the μ_i's satisfy* (3.1) *and $\lambda_i = \sum a_{ij}\mu_j$. Assume there are positive constants C_1, C_2, and r with $0 < r < 1$ such that for all i*

$$| a_{ij} | \leqslant C_1 r^{|i-j|} \sum_{|j-i|<k} | a_{ij} | \leqslant C_1 r^{|i-j|} C_2 . \tag{3.3}$$

Let $0 \leqslant a < \alpha < \beta < b \leqslant 1$ and let $f \in C[0, 1]$. Then, there exist constants K_1 and K_2 independent of f and Δ such that for $\bar{\Delta}$ sufficiently small

$$\| Pf - f \|_{L_\infty[\alpha,\beta]} \leqslant K_1 \{ \inf_{s\in S} \| f - s \|_{L_\infty[a,b]} + K_2 (\bar{\Delta})^k \inf_{s\in S} \| f - s \|_\infty \}. \tag{3.4}$$

Proof. Let $x \in [\alpha, \beta]$ and let i and m be such that $a \leqslant x_{i-m} \leqslant \alpha < x_i \leqslant x \leqslant x_{i+1} < \beta \leqslant x_{i+m+k} \leqslant b$. By (3.2) and (3.3) it suffices to show that for $\bar{\Delta}$ sufficiently small $r^m/(1 - r) \leqslant (\bar{\Delta})^k$. Now, $r^m/(1 - r) \leqslant h^k$ if and only if $m + \log_{1/r}(1 - r) \geqslant k \log_{1/r} h^{-1}$. Let $\delta = \min\{\alpha - a, b - \beta\}$; then,

$$\frac{1}{\bar{\Delta}} \leqslant \frac{2m + k}{x_{i+m+k} - x_{i-m}} \leqslant \frac{2m + k}{\delta} .$$

The desired result now follows easily.

It can now be seen that if each element of a sequence of projections $\{P_N\}$, $P_N: C[0, 1] \to S(\Delta_N)$, satisfies (3.1) and if there exist constants C_1, C_2, and r such that (3.3) is satisfied for all P_N, then we will have a local convergence theorem for this sequence.

A few observations on projections onto spline spaces are in order. First of all, if $P = \sum \lambda_i \otimes B_i$, then (cf. (1.2))

$$D_k^{-1} \max_i \| \lambda_i \| \leqslant \| P \| \leqslant \max_i \| \lambda_i \|.$$

If $\phi = \sum a_i \lambda_i$, then, obviously, $\| \phi \| \leqslant \max \| \lambda_i \| \sum | a_j |$. A converse inequality also holds:

$$\sum | a_j | = \left\langle \sum a_j \lambda_j , \sum \operatorname{sgn} a_i B_i \right\rangle \leqslant \| \phi \|.$$

Finally, the norm of an element $\phi = \sum a_i \lambda_i$ is essentially determined by its action on S. We make this precise in the following lemma.

LEMMA 3.2. *Let $P = \sum \lambda_i \otimes B_i$ be a projection onto S. Then, with $K = D_k \max_i \| \lambda_i \|$, we have for $\phi = \sum a_i \lambda_i$*

$$K^{-1} \| \phi \| \leqslant \sum |\langle \phi, B_i \rangle| \leqslant \| \phi \|. \tag{3.5}$$

Proof. Let $\phi = \sum a_i \lambda_i$ and let $\{c_i\}$ be arbitrary scalars. Then,

$$\left| \sum c_i \langle \phi, B_i \rangle \right| \leqslant \max_j | c_j | \sum_i \langle \phi, B_i \rangle | \leqslant D_k \sum_i |\langle \phi, B_i \rangle| \left\| \sum c_j B_j \right\|_\infty .$$

Consequently, by a theorem of Helly (cf. [1, p. 43]) there is a $\psi \in C^*$ such that $\langle \psi - \phi, B_i \rangle = 0 \ \forall i$ and $\| \psi \| \leqslant D_k \sum_i |\langle \phi, B_i \rangle|$. Now, $\| \phi \| = \| P\psi \| < \| P \| \| \psi \| \leqslant \max_i \| \lambda_i \| D_k \sum |\langle \phi, B_j \rangle|$. The other inequality is clear.

The above result has a converse. Namely, if (3.5) holds for some K, then $\| P \| \leqslant K$. To see this, note that for $\psi \in C^*$, $\| P^* \psi \| = \| \sum_i \langle \psi, B_i \rangle \lambda_i \| \leqslant K \sum_j |\langle \sum_i \langle \psi, B_i \rangle \lambda_i , B_j \rangle| = K \sum_j |\langle \psi, B_j \rangle| = K\langle \psi, \sum e_j B_j \rangle \leqslant K \| \psi \|$ for appropriate $e_j \in \{-1, 1\}$.

Corollary 3.5. *Suppose* $P = \sum \lambda_i \otimes B_i$ *is a projection. Let* $\{\phi_i\}$ *be a basis for* $\operatorname{span}\{\lambda_i\}$ *satisfying*

$$\Gamma^{-1} \sum | a_i | \leqslant \left\| \sum a_i \phi_i \right\| \leqslant \sum | a_i | \tag{3.6}$$

for all $\{a_i\}$. *Then, for* $\phi = \sum a_i \phi_i$,

$$(\Gamma K)^{-1} \sum | a_i | \leqslant \sum |\langle \phi, B_i \rangle|, \qquad \text{where } K \text{ is as in (3.5).}$$

Theorem 3.6. *Let* $P = \sum \lambda_i \otimes B_i$ *be the projection onto* S *determined by the functionals* $\{\phi_i\}$. *Assume that the* ϕ_i*'s satisfy* (3.1) *and* (3.6). *Further, assume that there is a constant* $\Lambda > 0$ *such that for all* i, *for all* $r \geqslant 1$, *and for all* $\{a_j\}$

$$\Lambda^{-1} \sum_{j=i}^{i+r} | a_j | \leqslant \sum_{j=i}^{i+r} \left| \left\langle \sum_{l=i}^{i+r} a_l \phi_l , B_j \right\rangle \right|. \tag{3.7}$$

Then, there exist constants K *and* r, $0 < r < 1$, *depending on only* Γ *and* $\| P \|$ *such that*

$$| a_{ij} | \leqslant K r^{|i-j|} \sum_{l=i}^{i+k-1} | a_{il} |. \tag{3.8}$$

Remark. From (3.5) and (3.6) it follows that (3.7) is equivalent to the assumption that the projections onto $\operatorname{span}\{B_i ,..., B_{i+r}\}$ determined by $\{\phi_i ,..., \phi_{i+r}\}$ are uniformly bounded.

Proof. Fix i. We prove the theorem for $j > i$. The case $j < i$ follows by symmetry. Let $j_0 \equiv i$ and $j_m = j_{m-1} + k = i + mk$. Choose $\epsilon_j^m \in \{-1, 1\}$ so that

$$\sum_{j \geqslant j_m} | a_{ij} | \leqslant \Lambda \left\langle \sum_{j \geqslant j_m} a_{ij} \phi_j , \sum_{j \geqslant j_m} \epsilon_j^m B_j \right\rangle.$$

Note that

$$\left\langle \sum_j a_{ij} \phi_j , \sum_{j \geqslant j_m} \epsilon_j^m B_j \right\rangle = \left\langle \lambda_i , \sum_{j \geqslant j_m} \epsilon_j^m B_j \right\rangle = 0$$

and that

$$\left\langle \sum_{j < j_{m-1}} a_{ij} \phi_j , \sum_{j \geqslant j_m} \epsilon_j^m B_j \right\rangle = 0.$$

Therefore,

$$\left\langle \sum_{j=j_{m-1}}^{j_m-1} a_{ij}\phi_j + \sum_{j\geqslant j_m} a_{ij}\phi_j \,,\ \sum_{j\geqslant j_m} \epsilon_j{}^m B_j \right\rangle = 0.$$

By (3.7) and (3.6)

$$\sum_{j\geqslant j_m} | a_{ij} | \leqslant \Lambda \sum_{j=j_{m-1}}^{j_m-1} | a_{ij} |.$$

Now, with $s_m = \sum_{j=j_{m-1}}^{j_m-1} | a_{ij} |$, we have $\sum_{m\geqslant 1} s_m = \sum_{j\geqslant j_0} | a_{ij} |$ and $\sum_{i>m} s_i \leqslant \Lambda s_m$ for $m \geqslant 0$. Therefore, by a lemma of Douglas, Dupont, and Wahlbin [8], it follows that $s_{m+1} \leqslant (\Lambda/(1+\Lambda))^{m-1} \Lambda s_1$. The desired result now follows easily.

The quantity $s_1 = \sum_{j=1}^{i+k-1} | a_{ij} |$ can be bounded as follows:

$$\sum_{j=i}^{i+k-1} | a_{ij} | \leqslant \sum_j | a_{ij} | \leqslant \Gamma \left\| \sum a_{ij}\phi_j \right\| = \Gamma \| \psi_i \| \leqslant \Gamma D_k \| P \|.$$

Corollary 3.7. *Let* $\{\Delta_N\}$ *be a sequence of partitions with* $\lim_{N\to\infty} \bar{\Delta}_N = 0$. *Let* $\{P_N\}$, P_N: $C[0, 1] \to S(\Delta_N) \equiv S_N$, *be a sequence of projections. Assume that there is a* $\Lambda_0 > 0$ *such that each* P_N *satisfies the hypothesis of Theorem* 3.6 *with* Λ_0 *in* (3.7). *Let* $0 \leqslant a < \alpha < \beta < b \leqslant 1$. *Then, there exists an integer* N_0 *and constants* K_1 *and* K_2 *such that for every* $f \in C[0, 1]$

$$\| P_N f - f \|_{L_\infty[\alpha,\beta]} \leqslant K_1 \{ \inf_{s\in S_N} \| f - s \|_{L_\infty[a,b]} + K_2 (\bar{\Delta}_N)^k \inf_{s\in S_N} \| f - s \|_\infty \}$$

for $N \geqslant N_0$.

Remarks. 1. Condition (3.1) may be relaxed to: There exists an $r \geqslant 0$ such that for all i, carr $\mu_i \subseteq \bigcup_{j=i-r}^{i+r}$ supp B_j and $\| \mu_i \| \leqslant 1$.

2. Theorem 3.6 can easily be extended to more general situations; all that is needed is that the spaces used for approximation have bases with properties similar to those of the normalized B-spline bases.

3. The condition that $\sum | a_i | \leqslant \Gamma \| a_i\phi_i \|$ for all $\{a_i\}$ for some $\Gamma > 0$ (3.5), can be replaced by: There is a $q < \infty$ and a $\Gamma > 0$ such that for all $\{a_i\}$, $\{\sum | a_i |^q\}^{1/q} \leqslant \Gamma \| \sum a_i\phi_i \|$. However, in many cases of interest, e.g., interpolation, the ϕ_i's have disjoint carriers so that (3.5) holds trivially.

4. Applications

Theorem 4.1. *Let* Δ *be given with* $x_i < x_{i+4}$ $\forall i$ *and let* P: $C[0, 1] \to$ $S(4, \Delta) \equiv S$ *be defined by* $Pf = s$ *if and only if* $f(\tau_i) = s(\tau_i)$, $-0 \leqslant i \leqslant N-1$, *where* $\tau_i = (x_{i+1} + x_{i+2} + x_{i+3})/3$. *If* $0 \leqslant a < \alpha < \beta < b \leqslant 1$ *and* $f \in C[0, 1]$, *then for* $\bar{\Delta}$ *sufficiently small*

$$\| D^j(Pf - f)\|_{L_\infty[\alpha,\beta]} \leqslant K\{\inf_{s\in S} \| f'' - s'' \|_{L_\infty[a,b]} + C(\bar{\Delta})^4 \inf_{s\in S} \| f - s \|_{L_\infty[0,1]}\}\cdot(\bar{\Delta})^{2-j}.$$

Proof. It suffices to verify (3.7). But, de Boor [6] has shown that for any sequence $\{x_i\}$, with $x_i < x_{i+4}$ $\forall i$, $\| P \| \leqslant 27$.

The local convergence theorems for the usual Type I cubic spline interpolation found by Kammerer and Reddien [9] required that the partitions be quasi-uniform. Later on, de Boor [5] showed that quasi-uniformity could be relaxed to a local mesh ratio restriction. The next result shows that if we restrict our attention to functions $f \in C^2[0, 1]$, then no mesh ratio restriction is necessary.

THEOREM 4.2. *Let* $\Delta: 0 = x_0 < x_1 < \cdots < x_N = 1$ *and let* $P: C^2[0, 1] \to S(4, \Delta) \equiv S$ *be defined by* $Pf = s$ *if and only if* $f(x_i) = s(x_i)$, $0 \leqslant i \leqslant N$, *and* $f'(x_i) = s'(x_i)$, $i = 0, N$. *Let* $0 \leqslant a < \alpha < \beta < b \leqslant 1$. *Then, for* $0 \leqslant j \leqslant 2$, *and* $\bar{\Delta}$ *sufficiently small,*

$$\| D^j(Pf - f)\|_{L_\infty[\alpha,\beta]} \leqslant K\{\inf_{s\in S} \| f - s \|_{L_\infty[a,b]} + C(\bar{\Delta})^4 \inf_{s\in S} \| f - s \|_{L_\infty[0,1]}\}.$$

Proof. This follows immediately from the fact that the least-squares projection $L: C[0, 1] \to S(2, \Delta)$ is bounded independent of Δ (cf. [3]).

In a similar way one can prove a mesh ratio free local convergence theorem for Type I quintic spline interpolation. In fact, in light of the results of [8], one can prove (for quasi-uniform partitions) similar local convergence theorems for Type I interpolation by any fixed odd-degree splines. It should be noted that proofs of the above results were "boundary condition free" and that, consequently, verification of (3.7) was trivial.

For our final application, we consider sequences of projections that are bounded in some L_p, $1 \leqslant p < \infty$. Let us first recall [4], that for $1 \leqslant p < \infty$ there is a constant $D_{k,p}$ such that for all sequences $\{a_i\}$

$$D_{k,p}^{-1} \left\{\sum | a_i |^p\right\}^{1/p} \leqslant \left\| \sum a_i B_{i,p} \right\|_p \leqslant \left\{\sum | a_i |^p\right\}^{1/p}, \tag{4.1}$$

where $B_{i,p} = \{k/(x_{i+k} - x_i)\}^{1/p} B_i$.

LEMMA 4.3. *Let* $P = \sum \lambda_i \otimes B_{i,p}$ *be a projection from* L_p *onto* S. *Then,*

(a) $D_{k,p}^{-1} \sup_{\|f\|_p=1}\{\sum |\langle\lambda_i, f\rangle|^p\}^{1/p} \leqslant \| P \| \leqslant \sup_{\|f\|_p=1}\{\sum |\langle\lambda_i, f\rangle|^p\}^{1/p}$;

(b) *for each* $\lambda \in \operatorname{span}\{\lambda_i\}$,

$$(\| P \| D_{k,p})^{-1} \| \lambda \|_q \leqslant \left\{\sum |\langle\lambda, B_{i,p}\rangle|^q\right\}^{1/q} \leqslant k^{1/q} \| \lambda \|_q ;$$

(c) *for each* $s \in S$,

$$(\| P \| D_{k,q})^{-1} \| s \|_p \leqslant \left\{\sum |\langle\lambda_i , s\rangle|^p\right\}^{1/p} \leqslant k^{1/p} \| s \|_p .$$

Here, $(1/q) + (1/p) = 1$.

Conversely, if for some $K > 0$, $K^{-1} \| \lambda \|_q \leqslant \{\sum |\langle\lambda, B_{i,p}\rangle|^q\}^{1/q}$ *hold for all* $\lambda \in \operatorname{span}\{\lambda_i\}$, *then,* $\| P \| \leqslant Kk^{1/q}$.

Proof. (a) follows from (4.1). The arguments used to prove (b) and (c) are simply modifications of the proof of Lemma 3.2. The converse is also easy.

The proof of the following result is similar to that of Theorem 3.6 so we omit it.

THEOREM 4.4. *Let* $1 \leqslant p < \infty$ *and let* $P = \sum \lambda_i \otimes B_{i,p}$ *be determined by linear functionals* $\{\phi_i\}$ *satisfying* (i) $\phi_i = 0$ *a.e. on* $[0, 1]\backslash(x_i , x_{i+k})$, (ii) $\| \phi_i \|_q \leqslant 1\ \forall i\ ((1/p) + 1/q = 1)$, *and* (iii) $\sum | a_j |^q \leqslant \gamma \| \sum a_j\phi_j \|_q^q$ *for some constant* $\gamma > 0$. *Further assume that there is a constant* $\Gamma > 0$ *so that for each i and for each* $r \geqslant k$ *the projection onto* $\operatorname{span}\{B_i ,..., B_{i+r}\}$ *determined by* $\{\phi_i ,..., \phi_{i+r}\}$ *is bounded in norm by* Γ. *Then,*

(a) *if* $2 \leqslant p < \infty$ *and* $\lambda_i = \sum a_{ij}\phi_j$,

$$| a_{ij} | \leqslant Kr^{|i-j|} \left\{\sum_{j=i}^{i+k} | a_{ij} |^q\right\}^{1/q} \qquad \text{for all } i,$$

where $K > 0$ *and* $0 < r < 1$ *depend only on* Γ;

(b) *if* $1 \leqslant p \leqslant 2$ *and* $C_i = \sum b_{ij}B_{j,p}$ *where* $\langle\phi_i , C_j\rangle = \delta_{ij}$,

$$| b_{ij} | \leqslant Kr^{|i-j|} \left\{\sum_{j=i}^{i+k} | a_{ij} |^p\right\}^{1/p} \qquad \text{for all } i,$$

where K and r are as in (a).

COROLLARY 4.5. *Let P satisfy the hypotheses of Theorem* 4.4. *Assume* Δ *is such that* $\max\{x_{i+k} - x_i\}/\min\{x_{i+k} - x_i\} \leqslant \sigma$. *Then, when viewed as an operator on* $C[0, 1]$, *the norm of P is bounded by* $(2\sigma K\gamma/(1 - r))\, D_{k,p} \| P \|_p$ *where K, r, and* γ *are as in Theorem* 4.4.

Proof. Assume $2 \leqslant p < \infty$. Then, $P = \sum h_i\lambda_i \otimes B_i$, where $h_i = \{k/x_{i+k-x_i}\}^{1/p}$, and $\| P \|_\infty \leqslant \max_i \| h_i\lambda_i \|_1$. But $\| \lambda_i \|_1 \leqslant \max_j \| \phi_j \|_1 \sum_j | a_{ij} |$. Hölder's inequality shows that $\| \phi_j \|_1 \leqslant (x_{j+k} - x_j)^{1/p} \| \phi_j \|_q \leqslant (x_{j+k} - x_j)^{1/p}$. Thus,

$$\| P \|_\infty \leqslant \sigma \max_i \sum_j | a_{ij} | \leqslant 2\sigma K \sum_j r^j \max_i \left\{\sum_{j=i}^{i+k} | a_{ij} |^q\right\}^{1/q}$$

Now, apply Lemma 4.3(b).

It is clear that a projection, $P = \sum \lambda_i \otimes B_{i,p}$, which is bounded on L_p is also bounded on L_∞; it is also clear that the L_∞ norm of P, in general, depends on more than just the L_p-norm of P. The above result yields that, with the hypotheses of Theorem 4.4 and with a global mesh ratio restriction, the L_∞ norm of P is of the same magnitude as the L_p norm. In some cases the mesh restriction might not be necessary. Finally, let us note that with $p = 2$ and $\phi_i = B_{i,2}$, we can recover the result of [8], that the least-squares projection can be bounded in terms of a global mesh ratio.

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